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INSTITUT FÜR INFORMATIONSSYSTEME

On the Behaviour of Permutation Entropy on Fractional Brownian Motion in a Multivariate Setting

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Abstract

- Investigation of qualitative behaviour of the fractional Brownian motion.
- Permutation Entropy quantifies complexity.
- Multi-Scale Permutation Entropy studies structures on different time scales.
- Numerous studies on the behaviour on fractional Brownian motion.
- Univariate context although many real-world challenges are multivariate.

Contribution

We investigate the behaviour of
(i) Permutation Entropy as well as
(ii) Multi-Scale Permutation Entropy
on fractional Brownian motion – each in the multivariate case.

Multivariate Fractional Brownian Motion

Definition

An m-multivariate process $((X^i(t))_{i=1}^m)_{t\in\mathbb{R}}$ is called *multivariate fractional Brownian motion* (m-mfBm or mfBm) $\mathrm{B}_H^m(t)$ with Hurst parameter $H \in \mathbb{R}^m$ with $H_i \in (0,1)$ for $i = 1, \ldots, m$ if it is

(i) Gaussian,
(ii) self-similar with Hurst parameter H and it has
(iii) stationary increments, i.e., $\mathrm{B}_H^m(t) - \mathrm{B}_H^m(s) \sim \mathrm{B}_H^m(t-s)$.

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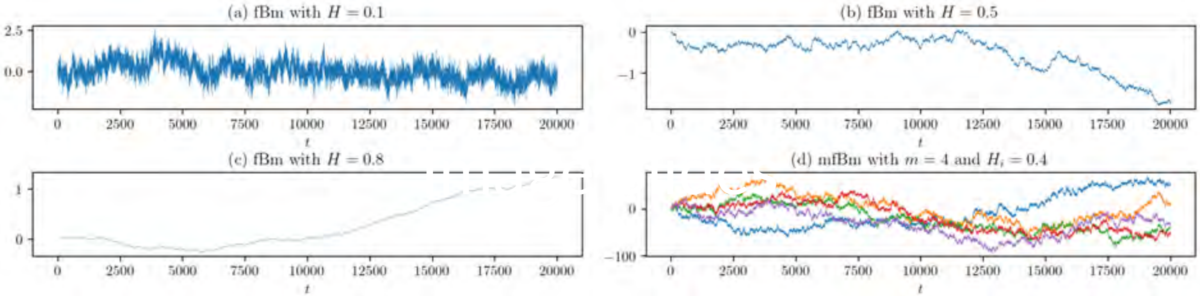
Mutivariate Fractional Brownian Motion

- $H < 1/2$: anti-persistence property and negatively correlated increments
- $H = 1/2$: ordinary Brownian motion
- $H > 1/2$: persistence property and positively correlated increments
- $H \to 1$: smoothness, less irregular and more trendy



Ordinal Pattern Symbolization

Ordinal patterns describe the total order between two or more neighbours, encoded by permutations.

$$(0,1,2),(2,1,0),(1,0,2),(2,0,1),(0,2,1),(1,2,0)$$

Figure: All possible ordinal patterns of order $d = 3$.

Definition

A vector $(x_1, \ldots, x_d) \in \mathbb{R}^d$ has *ordinal pattern* $(r_1, \ldots, r_d) \in \mathbb{N}^d$ of order $d \in \mathbb{N}$ if $x_{r_1} \geq \ldots \geq x_{r_d}$ and $r_{l-1} > r_l$ in the case $x_{r_{l-1}} = x_{r_l}$.

Ordinal Pattern in Time Series

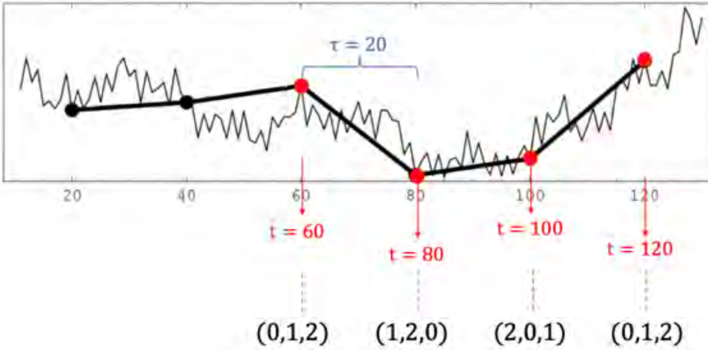


Figure: Ordinal pattern determination of order $d = 3$ and time delay $\tau = 20$ in a univariate time series.

Pooled Permutation Entropy

Calculate the Pooling matrix $P = \begin{pmatrix} p^{\tau,d}_{1,j_1} & \cdots & p^{\tau,d}_{1,j_{d!}} \\ \vdots & \ddots & \vdots \\ p^{\tau,d}_{m,1} & \cdots & p^{\tau,d}_{m,j_{d!}} \end{pmatrix} \in (0,1)^{m \times d!}$,

where $p^{\tau,d}_{ij}$ is the frequency of pattern j in variable i regarding a total count of $T - (d\tau - \tau) \cdot m$ ordinal patterns.

Pooled Permutation Entropy

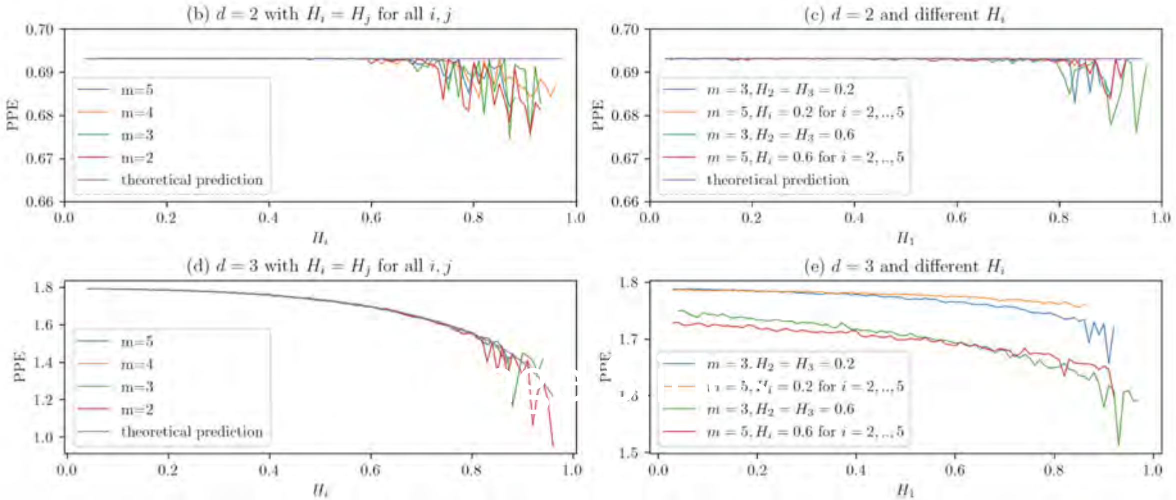
Definition

The *Pooled Permutation Entropy* (PPE) of a multivariate time series $\mathrm{X} = ((x_t^i)_{i=1}^m)_{t=1}^T$ is defined as the Permutation Entropy of the marginal relative frequencies $p_{.j}^{\tau,d} = \sum_{i=1}^m p_{ij}^{\tau,d}$ for $j = 1, \ldots, d!$ describing the distribution of the ordinal pattern and can be calculated as $\mathrm{PPE}_{d,\tau}(\mathrm{X}) = -\sum_j^{d!} p_{.j}^{\tau,d} \ln p_{.j}^{\tau,d}$.

We show that

- $\mathrm{PPE}_{2,\tau}(mfBm) = \ln \frac{1}{2}$ for all τ, m, H
- $\mathrm{PPE}_{3,\tau}(mfBm)$ is dependent of Hurst parameter H and independent of all τ, m

PPE on mfBm


(b) $d = 2$ with $H_i = H_j$ for all i, j
(c) $d = 2$ and different H_i
(d) $d = 3$ with $H_i = H_j$ for all i, j
(e) $d = 3$ and different H_i
PPE
H_i
H_1
m=5
m=4
m=3
m=2
theoretical prediction
$m = 3, H_2 = H_3 = 0.2$
$m = 5, H_i = 0.2$ for $i = 2, ..., 5$
$m = 3, H_2 = H_3 = 0.6$
$m = 5, H_i = 0.6$ for $i = 2, ..., 5$

Multiscale Permutation Entropy: Uncovering multiscale structural correlations to distinguish between randomness and complexity

Course-Graining

Per variable i in an multivariate time series, several consecutive time data points are averaged within a non-overlapping time window of the scaling length ϵ.
Each element of the coarse-grained time series $Y = ((y_{i,l}^{(\epsilon)})_{l=1}^{T/\epsilon})_{i=1}^{m}$ is calculated as:

$$y_{i,l}^{(\epsilon)} = \frac{1}{\epsilon} \sum_{t=(l-1)\epsilon+1}^{l\epsilon} x_{i,t} \tag{1}$$

for all $i = 1, ..., m$ and $1 \leq l \leq \frac{T}{\epsilon}$.

Multiscale Multivariate Permutation Entropy

Definition

The *Multivariate Multi-Scale Permutation Entropy* (MMSPE) of order $d \in \mathbb{N}$ and delay $\tau \in \mathbb{N}$ of a multivariate time series X is defined as PPE of its coarse-grained time series Y, that is $\mathrm{MMSPE}_{d,\tau,\epsilon}(\mathrm{X}) = \mathrm{PPE}_{d,\tau}(\mathrm{Y})$.

We show that

- $\mathrm{MMSPE}_{2,\tau}(mfBm) = \ln \frac{1}{2}$ for all τ, m, H
- $\mathrm{MMSPE}_{3,\tau}(mfBm)$ is dependent of Hurst parameter H and independent of all τ, m

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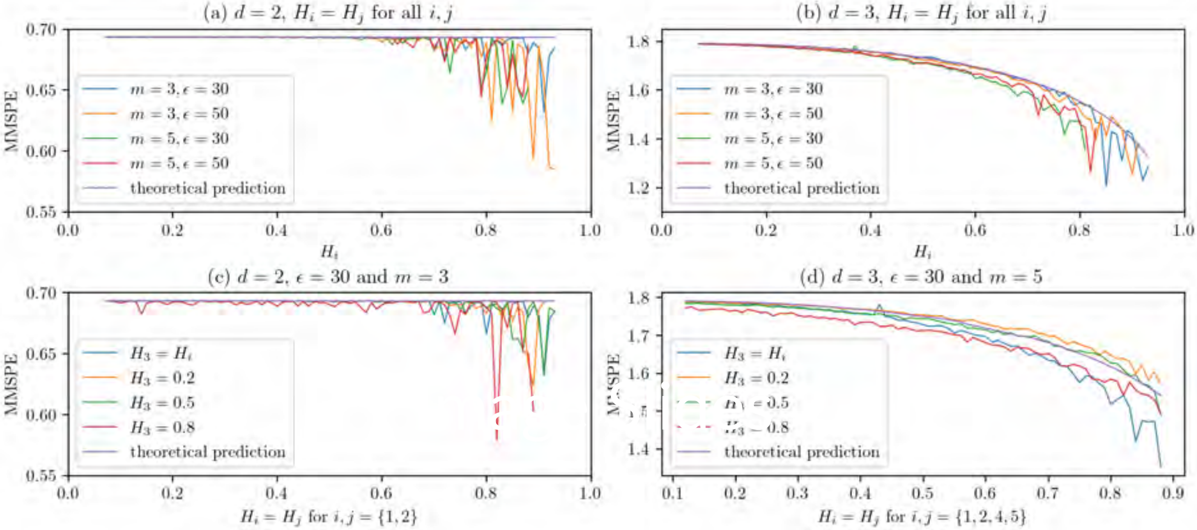
MMSPE on mfBm



Conclusion and Future Work

- Theoretical and experimental analysis of the behaviour of PPE and MMSPE on mfBm.
- PPE and MMSPE of order $d = 2$ on the mfBm is independent of all parameters, especially in the number of variables m.
- PPE and MMSPE of order $d = 3$ on mfBm is only dependent on the Hurst parameter $H \in \mathbb{R}^m$.
- PPE and MMSPE do not reveal relevant structures on mfBm with infinite length, although the long and short memory correlations generate a very complex behaviour.
- Further measurements are necessary.

Thank You

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