UNIVERSITÄT ZU LÜBECK
INSTITUT FÜR INFORMATIONSSYSTEME

On the Behaviour of Weighted Permutation Entropy on Fractional Brownian Motion in the Univariate and Multivariate Setting

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Abstract

- Investigation of qualitative behaviour of the fractional Brownian motion
- Permutation Entropy quantifies complexity
- Weighted Permutation Entropy takes amplitudes within time series into account
- Univariate context although many real-world challenges are multivariate

Contribution

(I) Introduction of multivariate weighted permutation entropy (MWPE)
(II) Investigation of the behaviour of weighted permutation entropy on both univariate and multivariate fractional Brownian motion

(Multivariate) Fractional Brownian Motion

Definition (Multivariate fractional Brownian motion [1])

An m-multivariate process $((X^i(t))_{i=1}^m)_{t\in\mathbb{R}}$ is called *multivariate fractional Brownian motion (mfBm)* with Hurst parameter $H = (H_1, \ldots, H_m)$, $H_i \in (0,1)$ for $i = 1, \ldots, m$, and denoted as $\mathrm{B}_H^m(t)$, if it is

- **Gaussian distributed**,
- **self-similar** with Hurst parameter H, i.e.,

$$(X_1(\quad t), \ldots, X_m(\quad t))_{t\in\mathrm{R}} \qquad (\quad ^{H_1}X_1(t), \ldots, \quad ^{H_m}X_m(t))_{t\in\mathrm{R}}$$

  for any $\quad > 0$, where $\quad$ denotes the equality of finite-dimensional distributions and it has

- **stationary** increments, i.e., $\mathrm{B}_H^m(t) \quad \mathrm{B}_H^m(s) \quad \mathrm{B}_H^m(t \quad s)$.

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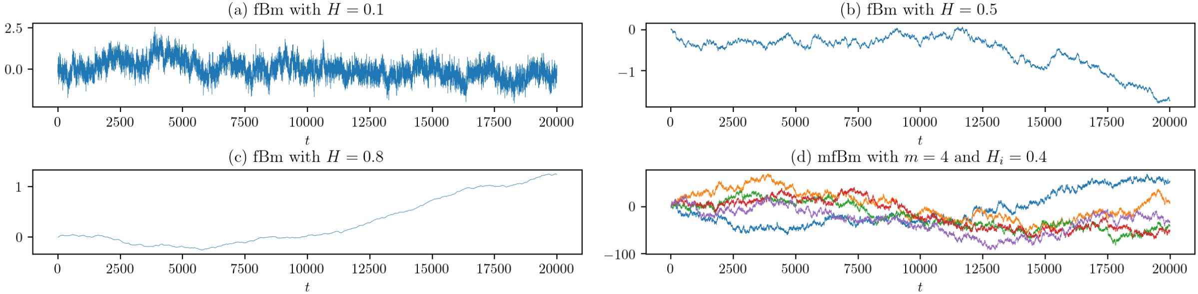
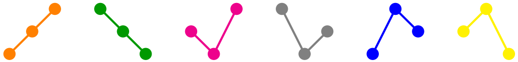
(Multivariate) Fractional Brownian Motion

- $H < 1/2$: anti-persistence property and negatively correlated increments
- $H = 1/2$: ordinary Brownian motion
- $H > 1/2$: persistence property and positively correlated increments
- $H \to 1$: smoothness, less irregular and more trendy



Ordinal Pattern Symbolization

Ordinal patterns describe the total order between two or more neighbours, encoded by permutations.



$(0,1,2)(2,1,0)(1,0,2)(2,0,1)(0,2,1)(1,2,0)$

Figure: All possible ordinal patterns of order $d = 3$.

Definition

A vector $(x_1,\ldots,x_d) \in \mathbb{R}^d$ has *ordinal pattern* $(r_1,\ldots,r_d) \in \mathbb{N}^d$ of order $d \in \mathbb{N}$ if $x_{r_1} \leq \ldots \leq x_{r_d}$ and $r_{l-1} > r_l$ in the case $x_{r_{l-1}} = x_{r_l}$.

Ordinal Pattern in Time Series

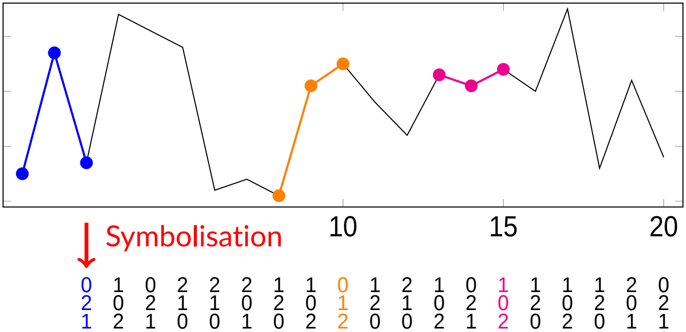


Figure: Ordinal pattern determination of order $d = 3$ and time delay $= 1$ at any time point t 2 $[d + 1; T]$.

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Permutation Entropy

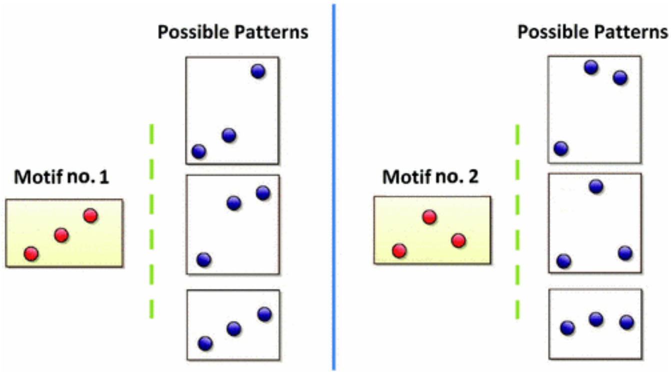
Definition (Permutation entropy [2])

The *permutation entropy (PE)* of order $d \in \mathbb{N}$ and delay $\tau \in \mathbb{N}$ of a univariate time series $\mathbf{x} = (x_t)_{t=1}^{T}$, $T \in \mathbb{N}$ is defined by

$$\mathrm{PE}_{d,\tau}(\mathbf{x}) = -\sum_{j=1}^{d!} p_j^{\tau,d} \ln p_j^{\tau,d}, \qquad (1)$$

where $p_j^{\tau,d}$ is the frequency of ordinal pattern j in the time series.

Different Amplitudes [3]



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Weighted Permutation Entropy: Differentiates between different patterns of a given motif in the amplitudes

Weighted Permutation Entropy

Definition (Weighted permutation entropy [3])

The *weighted permutation entropy (WPE)* of an univariate time series $\mathbf{x} = (x_t)_{t=1}^{T}$, $T \in \mathbb{N}$, with order $d \in \mathbb{N}$ and delay $\tau \in \mathbb{N}$ is defined as

$$\mathrm{WPE}_{d,\tau}(\mathbf{x}) = -\sum_{j}^{d!} wp_j^{\tau,d} \ln wp_j^{\tau,d} \qquad (2)$$

with

$$wp_j^{\tau,d} = \frac{\sum_{t \leq T} w_t \, \mathbb{1}[(x_{t-(d-1)\tau}, \ldots, x_{t-\tau}, x_t) \text{ has pattern } j]}{\sum_{t \leq T} w_t \, \mathbb{1}[(x_{t-(d-1)\tau}, \ldots, x_{t-\tau}, x_t)]}, \qquad (3)$$

where $w_t = \frac{1}{d}\sum_{k=1}^{d}(x_{t-(k-1)\tau} - \overline{x}_t^{d,\tau})^2$ is the empirical variance of the sub-sequence and $\overline{x}_t^{d,\tau}$ denotes the arithmetic mean that is $\overline{x}_t^{d,\tau} = \frac{1}{d}\sum_{k=1}^{d} x_{t-(k-1)\tau}$ and $\mathbb{1}[x] = 1$ if x, 0 otherwise.

WPE on fBm

Theorem (1)

The WPE of order $d = 2$ *for all delays* $\quad 2\ \mathrm{N}$ *on fBm* $B_H(t)$ *is given by*

$$\mathrm{WPE}_{2;}\,(B_H(t)) = \quad \ln(1{=}2): \tag{4}$$

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Proof

WPE differs from PE in that the ordinal patterns are weighted depending on their position t according to Eq. (3). For a weight w_t of order $d = 2$, i.e., of two time steps $x_{t \quad 1}$ and x_t, we have

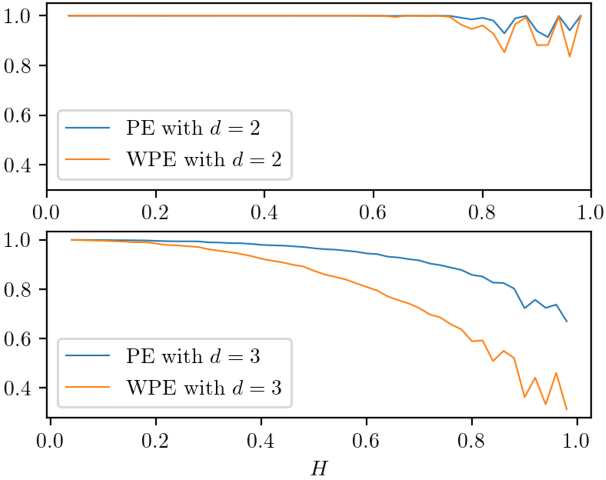
$$w_t = \frac{1}{2} \sum_{k=1}^{2} (x_{t \quad (k \quad 1)\tau} - \overline{x}_t^{2,\tau})^2 = \frac{1}{2} (x_t - x_{t \quad \tau})^2. \tag{5}$$

Since $x_t \sim B_H(t)$, we conclude from stationarity that

$$\frac{1}{2} (B_H(t) - B_H(t - \tau))^2 \sim \frac{1}{2} (B_H(\tau))^2 \tag{6}$$

with $\mathrm{Var}(B_H(\tau)) = \sigma^2 \tau^{2H}$ and $\sigma^2 = \mathrm{Var}(B_H(1)) = s^{2H}$. Consequently, the weights w_t are independently distributed from t, i.e., $w_t \sim \mathcal{N}(0, \frac{1}{2}(s\tau)^{2H})$. Considering the distribution of all possible realisations of fBm, we see from the use of the weights w_t from Eq. (3) in the calculation of WPE that it cancels out for a constant delay $\tau \in$ N.

WPE on fBm



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In many fields of applications, multivariate measurements are performed.

Multivariate Weighted Permutation Entropy

- For each variable $i = 1, \ldots, m$ and for each ordinal pattern $j = 1, \ldots, d!$ select all time steps in the time series $t \in [d \quad + 1, T]$, for which the variable time pair (i, t) has the ordinal pattern j.
- Add up the weights w_t, i.e., $\mathrm{w}_{ij} = \sum_{t=d \quad +1}^{T} w_t$ for all selected ordinal pattern vectors j and for each variable $i = 1, \ldots, m$. Note that the total count of weights w_t^i for each variable i is $\quad := T \quad (d \quad)$.
- Divide the weighted sum w_{ij} by the total sum of all $m \quad$ weights to obtain the weighted frequencies for every pattern j.
- Store the weighted frequencies in a *weighted pooling matrix* $P_w^{;d} \in \mathbb{R}^{m \quad d!}$, which reflects the weighted distribution of the ordinal patterns in the multivariate time series across its m variables.

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Multivariate Weighted Permutation Entropy

Definition (Multivariate weighted permutation entropy)

The *multivariate weighted permutation entropy (MWPE)* of a multivariate time series $X = ((x_t^i)_{i=1}^m)_{t=1}^T$ is defined as PE of the marginal weighted frequencies $P_{w_j}^{,d} = \sum_{i=1}^m P_{w_{ij}}^{,d}$ for $j = 1, \ldots, d!$ describing the distribution of the weighted ordinal pattern and can be calculated as

$$\text{MWPE}_{d,}(X) = \sum_{j}^{d!} P_{w_j}^{,d} \ln P_{w_j}^{,d}. \tag{7}$$

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MWPE in mfBm

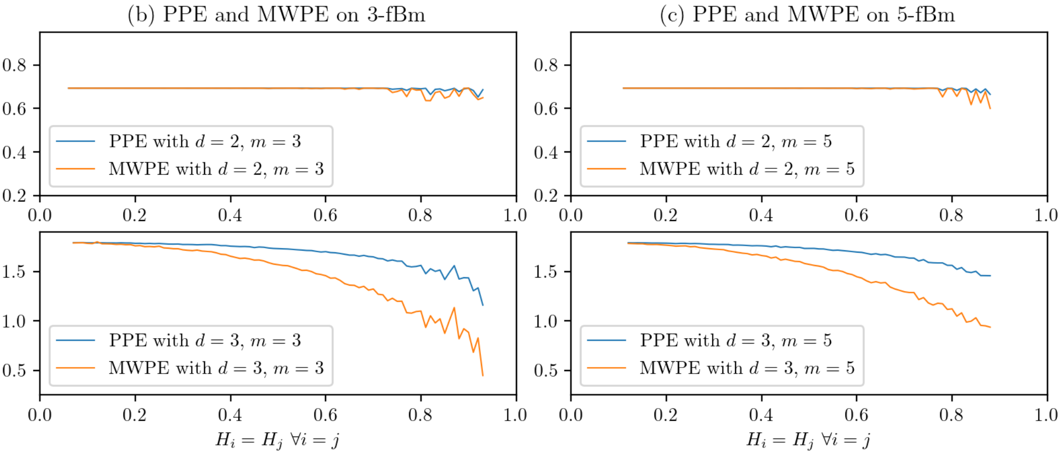
Theorem (2)

The MWPE of order $d = 2$ *for all delays* $2\ \mathrm{N}$ *on mfBm* $\mathrm{B}_H^m(t)$ *is given by*

$$\mathrm{MWPE}(\mathrm{B}_H^m(t)) = \quad \ln(1{=}2) \tag{8}$$

for all Hurst parameters H_i *and variable-dimensions* m*.*

MWPE on fBm



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Conclusion

- Definition of MWPE
- In case of order $d = 2$, we have shown that WPE or MWPE on fBm or mfBm is constant, i.e., there is no effect of weighting
- In case of order $d = 3$, we have shown that strictly ascending or strictly descending ordinal patterns $(0; 1; 2)$ or $(2; 1; 0)$ have greater weights and thus a greater impact on WPE and MWPE than the other four possible ordinal patterns

Future Work

- Application for the estimation of the Hurst parameter or in classical data analysis and machine learning tasks, such as outlier detection or time series classification
- Investigation of other orders $d > 3$ (*In Submission*)
- Consideration of cross-correlations of variables (*In Submission*)

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References

[1] Pierre-Olivier Amblard, Jean-François Coeurjolly, Frédéric Lavancier, and Anne Philippe. Basic properties of the Multivariate Fractional Brownian Motion. *Séminaires et congrès*, 28:65–87, 2013.

[2] Christoph Bandt and Bernd Pompe. Permutation Entropy: A Natural Complexity Measure for Time Series. *Physical Review Letters*, 88(17):174102, April 2002. Publisher: American Physical Society.

[3] Bilal Fadlallah, BD Chen, Andreas Keil, and Jose Principe. Weighted-permutation entropy: A complexity measure for time series incorporating amplitude information. *Physical review. E, Statistical, nonlinear, and soft matter physics*, 87:022911, 02 2013.